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SPECTRAL PROBLEMS FOR REGULAR CANONICAL DIRAC SYSTEMS

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Abstract. In this paper, several spectral results are achieved for regular Dirac system. The estimations are proved for solution in the Paley-Wiener space. The formula is obtained for the upper estimate of the boundary function in the second boundary condition.

Keywords: Dirac system, Paley-Wiener space, sampling theory.

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1. Introduction

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accessible to a wider audience. It treats in some depth the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states, and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory.

Note that, inverse problems for Dirac system had been investigated in woks [8, 11- 13, 14,]. It is well known [7] that two spectra uniquely determine the matrix-valued potential function. In particular, in work [11], eigenfunction expansions for one dimensional Dirac operators describing the motion of a particle in quantum mechanics are investigated.

Let L denote a matrix operator

$$L \equiv \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}, \quad p_{12}(x) \equiv p_{21}(x), \tag{1}$$

where the $p_{ik}(x)\ (i,k=1,2)$ are real functions which are defined and continuous on the interval $[0,\pi]$. Further, let $y(x,\lambda)$ denotes a two component vector function

$$y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}. \tag{2}$$

Then the equation

$$\left(B\frac{d}{dx}+L-\lambda I\right)y=0, \tag{3}$$

where λ is a parameter, and

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{4}$$

is equivalent to a system of two simultaneous first-order ordinary differential equations

$$\begin{aligned}&\frac{dy_2}{dx}+p_{11}(x)y_1+p_{12}(x)y_2=\lambda y_1,\\ -&\frac{dy_1}{dx}+p_{21}(x)y_1+p_{22}(x)y_2=\lambda y_2,\end{aligned} \tag{5}$$

For the case in which

$$p_{12}(x)=p_{21}(x)\equiv 0,\ p_{11}(x)=V(x)+m,\ p_{22}(x)=V(x)-m,$$

where $V(x)$ is a potential function, and m is the mass of a particle, the system (5) is known in relativistic quantum theory as a stationary one-dimensional Dirac system. For the case in which

$$\begin{aligned}&p_{12}(x)=p_{21}(x)=0\\ &p_{11}(x)=V(x)+m=p(x)\\ &p_{22}(x)=V(x)-m=r(x),\end{aligned}$$

we obtain the following system called first canonic form of Dirac operator.

$$\begin{aligned}&y_2'-\{\lambda-p(x)\}y_1=0,\\ &y_1'+\{\lambda-r(x)\}y_2=0,\end{aligned} \tag{6}$$

Thus, let us consider the boundary-value problem for the system (5), reducing it to the canonical form:

$$y_2'-\{\lambda-p(x)\}y_1=0,\quad y_1'+\{\lambda-r(x)\}y_2=0, \tag{7}$$

with the following boundary conditions

$$y_2(0)\cos\alpha+y_1(0)\sin\alpha=0, \tag{8}$$

$$y_2(\pi)\cos\beta+y_1(\pi)\sin\beta=0. \tag{9}$$

We will assume that the functions $p(x)$ and $r(x)$ are continuous on the interval $[0,\pi]$.

Let us denote by

$$y(x,\lambda)=\begin{pmatrix}y_1(x,\lambda)\\ y_2(x,\lambda)\end{pmatrix} \tag{10}$$

the solution of the system (7) satisfying the initial conditions

$$y_1(0,\lambda)=\cos\alpha,\quad y_2(0,\lambda)=-\sin\alpha. \tag{11}$$

The function $y(x,\lambda)$ obviously satisfies the boundary condition (8). Let us consider the problem (7), (11) for $p(x)=r(x)\equiv 0$. As is not difficult to see, in this case

$$y_1(x,\lambda)=\cos(\lambda x-\alpha),\quad y_2(x,\lambda)=\sin(\lambda x-\alpha). \tag{12}$$

Functions of $\alpha(x)$ and $\beta(x)$ have the expressions

$$\begin{aligned}\alpha(x)&=\cos\left\{\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]d\tau\right\},\\ \beta(x)&=\sin\left\{\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]d\tau\right\}.\end{aligned} \tag{13}$$

For the solution

$$y(x,\lambda)=\begin{pmatrix}y_1(x,\lambda)\\ y_2(x,\lambda)\end{pmatrix}$$

of the problem (7), (11) we are possessed of the following formulas [8]:

$$\begin{aligned}y_1(x,\lambda)=&\,\alpha(x)\cos(\lambda x-\alpha)+\beta(x)\sin(\lambda x-\alpha)+\\ &+\int_0^x\left\{P(x,s)\cos(\lambda s-\alpha)+R(x,s)\sin(\lambda s-\alpha)\right\}ds\end{aligned} \tag{14}$$

$$\begin{aligned}y_2(x,\lambda)=&\,\alpha(x)\sin(\lambda x-\alpha)-\beta(x)\cos(\lambda x-\alpha)+\\ &+\int_0^x\left\{Q(x,s)\cos(\lambda s-\alpha)+H(x,s)\sin(\lambda s-\alpha)\right\}ds\end{aligned} \tag{15}$$

or, inserting the values for $\alpha(x)$ and $\beta(x)$ from (13),

$$y_1(x,\lambda)=\cos\left\{\xi(x,\lambda)-\alpha\right\}+\int_0^x\left\{P(x,s)\cos(\lambda s-\alpha)+R(x,s)\sin(\lambda s-\alpha)\right\}ds,$$

$$y_2(x,\lambda)=\sin\left\{\xi(x,\lambda)-\alpha\right\}+\int_0^x\left\{Q(x,s)\cos(\lambda s-\alpha)+H(x,s)\sin(\lambda s-\alpha)\right\}ds,$$

where

$$\xi(x,\lambda)=\lambda x-\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]d\tau.$$

Consider the Dirac system with more general separable boundary conditions

$$\begin{aligned}&y_2'-\left\{\lambda-p(x)\right\}y_1=0,\quad y_1'+\left\{\lambda-r(x)\right\}y_2=0,\\ &a_{11}y_2(0,\lambda)-a_{12}y_1(0,\lambda)=0,\end{aligned} \tag{16}$$

$$a_{21}y_2(\pi,\lambda)+a_{22}y_1(\pi,\lambda)=0.$$

So, let $\lambda=\mu^2$ and $y_1(x,\mu^2)$, $y_2(x,\mu^2)$ denote the solutions of the initial value problem

$$y_2'-\{\mu^2-p(x)\}y_1=0,\quad y_1'+\{\mu^2-r(x)\}y_2=0,$$

$$y_1(0,\mu^2)=a_{11},\quad y_2(0,\mu^2)=a_{12}.$$

The eigenvalues of (1.16) are the square of the zeroes of the boundary function

$$B(\mu):=a_{21}y_2(\pi,\mu^2)+a_{22}y_1(\pi,\mu^2). \tag{17}$$

In the Dirichlet case, this boundary function is an entire function of μ of order 1 and type π and is square integrable on the real line. Thus, it belongs to the Paley-Wiener space.

$$PW_\pi=\left\{f\ entire,\ |f(\mu)|\le Ce^{\pi|I_m\mu|},\int_R |f(\mu)|^2 d\mu<\infty\right\}.$$

2. Main results

Let

$$v_{11}(x,\mu)=y_1(x,\mu^2)-\alpha(x)\cos(\mu^2x-\alpha)-\beta(x)\sin(\mu^2x-\alpha) \tag{18}$$

and

$$v_{21}(x,\mu)=y_2(x,\mu^2)-\alpha(x)\sin(\mu^2x-\alpha)+\beta(x)\cos(\mu^2x-\alpha) \tag{19}$$

In the following, we shall make use of the estimates [10],

$$|\cos u|\le e^{|\mathrm{Im}u|},\quad |\sin u|\le c_0e^{|\mathrm{Im}u|}, \tag{20}$$

where c_0 is some constant (we may take $c_0=1.72$ for numerical purposes).

Define the constants

$$c_1=\int_0^\pi \max_{0\le x\le\pi}|P(x,s)|ds\ ,\ c_2=\int_0^\pi \max_{0\le x\le\pi}|R(x,s)|ds,$$

$$c_3=\int_0^\pi \max_{0\le x\le\pi}|Q(x,s)|ds\ ,\ c_4=\int_0^\pi \max_{0\le x\le\pi}|H(x,s)|ds, \tag{21}$$

$$c_5=|a_{21}|(c_1+c_0c_2)\ ,\qquad c_6=|a_{22}|(c_3+c_0c_4).$$

we claim the subsequent results.

Theorem 1. $v_{11}(x,\mu),v_{21}(x,\mu)\in PW_x$ are functions of μ for each x and the following estimates hold:

$$\left|v_{11}(x,\mu)\right| \le \left(c_1 + c_0 c_2\right) e^{x\left|\operatorname{Im}\mu^2\right|}$$
$$\left|v_{21}(x,\mu)\right| \le \left(c_3 + c_0 c_4\right) e^{x\left|\operatorname{Im}\mu^2\right|} \tag{22}$$

Proof: In the first instance

$$\left|v_{11}(x,\mu)\right| \le \int_0^x \left|P(x,s)\right| \left|\cos\left(\mu^2 s - \alpha\right)\right| ds + \int_0^x \left|R(x,s)\right| \left|\sin\left(\mu^2 s - \alpha\right)\right| ds \tag{23}$$

Hence,

$$\left|v_{11}(x,\mu)\right| \le \int_0^x \left|P(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds + c_0 \int_0^x \left|R(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|} \left(\int_0^x \left|P(x,s)\right| ds + c_0 \int_0^x \left|R(x,s)\right| ds \right)$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|} \left(\int_0^\pi \max_{0\le x\le\pi} \left(\left|P(x,s)\right|\right) ds + c_0 \int_0^\pi \max_{0\le x\le\pi} \left(\left|R(x,s)\right|\right) ds \right)$$

from which we acquire

$$\left|v_{11}(x,\mu)\right| \le e^{x\left|\operatorname{Im}\mu^2\right|} \left(c_1 + c_0 c_2\right)$$

Therefore, $v_{11}(x,\mu)$ is entirely of type x order 1 and square integrable on the real line as a function of μ for each x, and satisfy the estimate.

In addition to,

$$\left|v_{21}(x,\mu)\right| \le \int_0^x \left|Q(x,s)\right| \left|\cos\left(\mu^2 s - \alpha\right)\right| ds + \int_0^x \left|H(x,s)\right| \left|\sin\left(\mu^2 s - \alpha\right)\right| ds \tag{24}$$

and

$$\left|v_{21}(x,\mu)\right| \le \int_0^x \left|Q(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds + c_0 \int_0^x \left|H(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|} \left(\int_0^x \left|Q(x,s)\right| ds + c_0 \int_0^x \left|H(x,s)\right| ds \right)$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|} \left(\int_0^\pi \max_{0\le x\le\pi} \left(\left|Q(x,s)\right|\right) ds + c_0 \int_0^\pi \max_{0\le x\le\pi} \left(\left|H(x,s)\right|\right) ds \right)$$

from which we obtain

$$\left|v_{21}(x,\mu)\right| \le e^{x\left|\operatorname{Im}\mu^2\right|} \left(c_3 + c_0 c_4\right)$$

Thus, we have proved the estimates. Therefore, $v_{21}(x,\mu)$ is entirely of type x order 1 and square integrable on the real line as a function of μ for each x.

Theorem 2. $v_{12}(x,\mu), v_{22}(x,\mu) \in PW_x$ are functions of μ for each x and the following estimates hold:

$$\begin{aligned} \left|v_{12}(x,\mu)\right| &\le c_0 c_2 e^{x\left|\operatorname{Im}\mu^2\right|} \\ \left|v_{22}(x,\mu)\right| &\le c_0 c_4 e^{x\left|\operatorname{Im}\mu^2\right|} \end{aligned} \qquad (25)$$

Proof:

$$v_{12}(x,\mu) = \int_0^x \left\{P(x,s)\cos\left(\mu^2 s-\alpha\right) + R(x,s)\sin\left(\mu^2 s-\alpha\right)\right\}ds -$$

$$-\int_0^x \left\{P(x,s)\cos\left(\mu^2 s-\alpha\right)\right\}ds$$

we obtain at once,

$$\begin{aligned} \left|v_{12}(x,\mu)\right| &\le \int_0^x \left|R(x,s)\right|\left|\sin\left(\mu^2 s-\alpha\right)\right| ds \\ &\le \int_0^x c_0 \left|R(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds \\ &\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_0\int_0^x \left|R(x,s)\right| ds\right) \\ &\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_0\int_0^\pi \max_{0\le x\le\pi}\left(\left|R(x,s)\right|\right) ds\right) \end{aligned}$$

from which we get

$$\left|v_{12}(x,\mu)\right| \le \left(c_0 c_2\right) e^{x\left|\operatorname{Im}\mu^2\right|}$$

However,

$$v_{22}(x,\mu) = \int_0^x \left\{Q(x,s)\cos\left(\mu^2 s-\alpha\right) + H(x,s)\sin\left(\mu^2 s-\alpha\right)\right\}ds -$$

$$-\int_0^x \left\{Q(x,s)\cos\left(\mu^2 s-\alpha\right)\right\}ds$$

we obtain

$$
\begin{aligned}
\left|v_{22}(x,\mu)\right| &\le \int_0^x \left|H(x,s)\right|\left|\sin\left(\mu^2 s-\alpha\right)\right| ds \\
&\le \int_0^x c_0\left|H(x,s)\right| e^{\left|\operatorname{Im}\left(\mu^2 s\right)\right|} ds \\
&\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_0\int_0^x \left|H(x,s)\right| ds\right) \\
&\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_0\int_0^\pi \max_{0\le x\le\pi}\left(\left|H(x,s)\right|\right) ds\right) \\
&= \left(c_0 c_4\right) e^{x\left|\operatorname{Im}\mu^2\right|}
\end{aligned}
$$

so, estimates are proved. Hence, functions $v_{12}(x,\mu), v_{22}(x,\mu)$ are entirely of type x order 1 and square integrable on the real line as a function of μ for each x.

The boundary function (characteristic equation) $B(\mu)$ is not necessarily in PW_π as in the Dirichlet case. However, we have the following theorem.

Theorem 3. $\tilde{B}(\pi,\mu)=a_{21}v_{11}(\pi,\mu)+a_{22}v_{21}(\pi,\mu)\in PW_\pi$ is a function of μ and the following estimate holds:

$$\left|\tilde{B}(\pi,\mu)\right| \le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_5+c_6\right) \tag{26}$$

Proof: Primarily, we have

$$\left|\tilde{B}(x,\mu)\right| \le \left|a_{21}\right|\left|v_{11}(x,\mu)\right|+\left|a_{22}\right|\left|v_{21}(x,\mu)\right| \tag{27}$$

from which we attain

$$
\begin{aligned}
&\le \left|a_{21}\right|\left(c_1+c_0c_2\right)e^{x\left|\operatorname{Im}\mu^2\right|}+\left|a_{22}\right|\left(c_3+c_0c_4\right)e^{x\left|\operatorname{Im}\mu^2\right|} \\
&\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_5+c_6\right)
\end{aligned}
$$

$\left|\tilde{B}(\pi,\mu)\right|$ is easily seen from above mentioned inequality. Hence, the following theorem is applicable.

Theorem 4. Let $f\in PW_\pi$, then

$$f(\mu)=\sum_{k=-\infty}^{\infty} f(k)\frac{\sin\pi(\mu-k)}{\pi(\mu-k)}, \tag{28}$$

where the series converges uniformly on compact set and also in $L^2_{d\mu}$ [10].

Let $\tilde{B}_N(\pi,\mu)$ denote the truncation of $\tilde{B}(\pi,\mu)$

$$\tilde{B}_N(\pi,\mu)=\sum_{k=-N}^{N}\tilde{B}(\pi,k)\frac{\sin\pi(\mu-k)}{\pi(\mu-k)}, \quad (29)$$

and $B_N(\pi,\mu)$ the corresponding approximation to $B(\pi,\mu)$.

3. Conclusion

In this paper, we examined Dirac system and succeeded in performing our approach for regular canonical Dirac systems. The approach is based on the well established technique: Shannon's sampling theorem. Thus, we obtained satisfactory results by using the Paley-Wiener spaces.

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Requlyar kanonik Dirak sistemi üçün spektral məsələlər

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XÜLASƏ

Məqalədə requlyar Dirak sistemi üçün bəzi spektral nəticələr alınmışdır. Paley-Viner fəzasında həllərin qiymətləndirilməsi isbat edilmiş, ikinci sərhəd sərtində sərhəd funksiyasının yuxarıdan qiymətləndirilməsi üçün düstur isbat edilmişdir.

Açar sözlər: Dirak sistemi, Paley-Viner fəzaları, diskretləşdirmə nəzəriyyəsi.

Спектральные задачи для регулярных канонических систем Дирака

Э. С. Панахов, М. Бабаоглы

РЕЗЮМЕ

В этой статье получены спектральные результаты для регулярной систем Дирака. Доказаны оценки для решения в пространстве Палея-Винера. Доказана формула для верхней оценки граничной функции во втором краевом условии.

Ключевые слова: система Дирака, пространства Палея-Винера, теория дискретизации.